

The SIR epidemic model from a PDE point of view

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Abstract

We present a derivation of the classical SIR model through a mean-field approximation from a discrete version of SIR. We then obtain a hyperbolic forward Kolmogorov equation, and show that its projected characteristics recover the standard SIR model. Moreover, we show that the long time limit of the evolution will be a Dirac measure. The exact position will depend on the well-know R_0 parameter, and it will be supported on the corresponding stable SIR equilibrium.

1. Introduction

A very fruitful modeling paradigm in epidemiology is the so-called compartmental models, with dynamics governed by mass-action laws. Most classical epidemiological models are of this type, and this has led to a number of both quantitative and qualitative predictions in the disease dynamics [2]. More recently, there is a growing interest in discrete, agent-based, models [7]. See also [6] for a comparison between different models. In many cases, these models are thought to be more realistic, and able to capture important dynamical features that are not present in the continuous models.

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Here, we follow the ideas in [3], to study the the large-population regime of the discrete dynamics. In this way, we obtain a Hyperbolic Forward Kolmogorov equation for the probability density evolution. It is well known, through the method of characteristics, that there is a strong linkage between solutions to first-order Partial Differential Equations (PDEs) and systems of Ordinary Differential Equations (ODEs). Thus, its not entirely surprising that projected characteristics of this PDE will be related to the classical SIR (Susceptible-Infected-Removed) ODE model.

This modeling through PDEs has some advantages, in particular, it allows the introduction of higher-order effects, like, for example, stochasticity, adding a second-order term to the equation. On the other hand, if we consider a discrete model in population dynamics and consider its limit of large population (under suitable condition) we naturally obtain a PDE for $p(t, x)$, the probability density to find the population at state x at time t . The ODE can then be obtained as the hyperbolic limit of the PDE, or, alternatively as the initial dynamics of the PDE. In short, this means that the dynamics of the discrete population can be approximated for short times and large population by a certain ODE — the derivation of this ODE requires an introduction of intermediate models, a stochastic differential equation or a partial differential one.

The ODE approach can be seen in [1], while the PDE modeling was the subject of a previous work from the authors, where the replicator equation was obtained as the limit of the finite-population discrete Moran process [3]. The resulting equation is of singular type and required a specific analysis of its behavior [4].

In this work, we will study in a certain level of detail the SIR epidemic model from the PDE point of view¹. This is one of the most elementary and well studied model in mathematical epidemiology. See [5, 2].

In particular, we shall prove that the solution of the SIR-hyperbolic-PDE obtained as a first order expansion in the inverse of the population size from the discrete-SIR converges when $t \rightarrow \infty$ to a Dirac-delta measure supported at the unique stable equilibrium of the SIR-ODE. Moreover (and this will be proved in a forthcoming work) the solution of the SIR-hyperbolic

¹The expression SIR appears in the literature in two different context: one as a discrete evolutionary system, used in general in computer simulations; the second as an ODE system. We hope that all these different meanings are clear from the context.

is approximated by the SIR-ODE in all time scales and approximate the discrete-SIR for short time scales. This provides a framework to unify all these descriptions. In a forthcoming work, we will also introduce the SIR-parabolic-PDE which has the inverse behavior (approximate the discrete-SIR for all time scales and the ODE-SIR for short times).

2. Discrete and continuous SIR models

Consider a discrete SIR model, i.e., consider a fixed size population of N individuals, each one in one of the three states: n individuals **S**usceptible (i.e, individuals with no imunity), m individuals **I**nfectious (i.e., individual currently infected by a given infectious disease and able to transmit it to the susceptibles) and $N - n - m$ **R**emoved (after infection individuals have a temporary imunity and then are removed from the dynamics; after certain time they become susceptible again). This is a very simple model for non lethal diseases transmitted by contact, e.g., normal influenza. At each time step of size $\Delta t > 0$ we select one individual at random:

- If it is **S**, then it changes to **I** with probability proportional to the fraction of **I** in the remainder, $\alpha m / (N - 1)$;
- If it is **I**, then it changes to **R** with constant probability β ;
- If it is **R**, then it changes to **S** with constant probability γ .

This can be summarized in the following diagram:

$$\mathbf{S} + \mathbf{I} \xrightarrow{\alpha} \mathbf{I} + \mathbf{I}, \quad \mathbf{I} \xrightarrow{\beta} \mathbf{R}, \quad \mathbf{R} \xrightarrow{\gamma} \mathbf{S}.$$

This model is also called SIRS, and when $\gamma = 0$ we recover the classical SIR model. For simplicity, however, we will call it the SIR model in this work and all results presented here include the case $\gamma = 0$.

Constants α , β and γ depend, in principle, in N and Δt . As we are interested in the limit behavior when $N \rightarrow \infty$, $\Delta t \rightarrow 0$ we will assume the following *scaling* relations

$$\lim_{N \rightarrow \infty, \Delta t \rightarrow 0} \frac{\alpha}{N \Delta t} = a, \quad \lim_{N \rightarrow \infty, \Delta t \rightarrow 0} \frac{\beta}{N \Delta t} = b, \quad \lim_{N \rightarrow \infty, \Delta t \rightarrow 0} \frac{\gamma}{N \Delta t} = c,$$

with $ab \neq 0$. For further informations on scalings, see [3].

Let $P_{(N,\Delta t)}(t, n, m)$ be the probability that at time t we have n susceptible, m infected and $N - n - m$ removed, where the total population N is constant and the time step is given by $\Delta t > 0$. Therefore

$$\begin{aligned} P_{(N,\Delta t)}(t + \Delta t, n, m) = & \alpha \frac{(n+1)(m-1)}{N(N-1)} P_{(N,\Delta t)}(t, n+1, m-1) \\ & + \beta \frac{m+1}{N} P_{(N,\Delta t)}(t, n, m+1) + \gamma \frac{N-n-m+1}{N} P_{(N,\Delta t)}(t, n-1, m) \\ & + \left[\frac{n}{N} \left(1 - \alpha \frac{m}{N-1} \right) + \frac{m}{N} (1 - \beta) + \frac{N-n-m}{N} (1 - \gamma) \right] P_{(N,\Delta t)}(t, n, m) . \end{aligned}$$

Now, define $x = n/N$, $y = m/N$ and $p(t, x, y) = P(t, xN, yN; N)$. Then, using $p(t, x, y) = p$ and keeping terms until order $1/N$:

$$\begin{aligned} p(t + \Delta t, x, y) = & \alpha \frac{(x + \frac{1}{N})(y - \frac{1}{N})}{(1 - \frac{1}{N})} p\left(t, x + \frac{1}{N}, y - \frac{1}{N}\right) \\ & + \beta \left(y + \frac{1}{N}\right) p\left(t, x, y + \frac{1}{N}\right) + \gamma \left(1 - x - y + \frac{1}{N}\right) p\left(t, x - \frac{1}{N}, y\right) \\ & + \left(x \left(1 - \frac{\alpha y}{1 - \frac{1}{N}}\right) + y(1 - \beta) + (1 - x - y)(1 - \gamma)\right) p(t, x, y) \\ \approx & p + \frac{1}{N} [(\alpha(y - x) + \beta + \gamma)p + (\alpha xy - \gamma(1 - x - y)) \partial_x p + (\beta y - \alpha xy) \partial_y p] \\ = & p + \frac{1}{N} [\partial_x ((\alpha xy - \gamma(1 - x - y))p) + \partial_y ((\beta - \alpha x)yp)] \end{aligned}$$

Finally,

$$\partial_t p = \partial_x ((axy - c(1 - x - y))p) + \partial_y ((b - ax)yp) . \quad (1)$$

subject to probability conservation, i.e.,

$$\frac{d}{dt} \int_0^1 p(t, x) dx = 0. \quad (2)$$

3. The SIR model as a transport problem

Let $\mathbf{x} = (x, y)$, and let $\Phi_t(\mathbf{x})$ be the flow map associated to the SIR system

$$\begin{aligned} \dot{X} &= c(1 - X - Y) - aXY , \\ \dot{Y} &= (aX - b)Y , \end{aligned}$$

then, if $p_0(\mathbf{x})$ is $C^1(R^2)$ function, and $p(t, x, y) = e^{Q(\mathbf{x}) - Q(\Phi_{-t}(\mathbf{x}))} p_0(\Phi_{-t}(\mathbf{x}))$, with $\mathbf{x} = (x, y)$, and Q satisfies

$$F \cdot \nabla Q = -\nabla \cdot F,$$

where F denotes the right hand side of the SIR system. Let \mathcal{S} denote the unit simplex in \mathbb{R}^2 .

Fix $\mathbf{x}_0 \in \mathcal{S}$. Then, we have that

$$e^{-Q(\phi_t(\mathbf{x}_0))} p(t, \phi_t(\mathbf{x}_0)) = e^{Q(\mathbf{x}_0)} p_0(\mathbf{x}_0).$$

Hence,

$$\begin{aligned} 0 &= e^{Q(\phi_t(\mathbf{x}_0))} \frac{d}{dt} [e^{-Q(\phi_t(\mathbf{x}_0))} p(t, \phi_t(\mathbf{x}_0))] \\ &= -F(\phi_t(\mathbf{x}_0)) \cdot \nabla Q(\phi_t(\mathbf{x}_0)) p(t, \phi_t(\mathbf{x}_0)) + \partial_t p(t, \phi_t(\mathbf{x}_0)) + F(\phi_t(\mathbf{x}_0)) \cdot \nabla p(t, \phi_t(\mathbf{x}_0)) \\ &= \nabla \cdot Fp + F \cdot \nabla p + \partial_t p \\ &= \partial_t p + \nabla \cdot (pF). \end{aligned}$$

Thus, the SIR system are the characteristics of (1), and the probability density should be transported along them. We now make this calculation more precise.

Definition 1. Let $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, be a Lipschitz vector field, where U is an open set, and let $\Omega \subset U$ be compact. We say that Ω is regularly attracting for F , if there is an open set V with a piecewise smooth boundary and $\Omega \subset V \subset \bar{V} \subset U$, such that, if Φ_t denotes the flow by F restricted to V , then we have that $\omega(V) \subset \bar{\Omega}$.

Theorem 1. Let Ω be a domain with piecewise smooth boundary $\partial\Omega$. Let $F : U \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$ be Lipschitz, with $\Omega \subset U$ being a regularly attracting set for F . Let $p_0 \in L^1(\Omega)$ be nonnegative. Then the equation

$$\partial_t p + \nabla \cdot (pF) = 0, \quad p(0, x) = p_0(x) \tag{3}$$

has a unique solution

$$p(t, \mathbf{x}) = e^{Q(\mathbf{x}) - Q(\Phi_{-t}(\mathbf{x}))} p_0(\Phi_{-t}(\mathbf{x}))$$

Moreover, p is nonnegative, and $\text{supp}(p(t, \cdot)) \subset \Omega$.

In addition, if $\text{supp}(p_0) \subset \Omega$, then

$$\frac{d}{dt} \int_{\Omega} p(t, \mathbf{x}) d\mathbf{x} = 0. \tag{4}$$

Proof. Existence can be shown as follows by considering the weak formulation, Let $W = [0, \infty) \times \Omega$, and let $\psi \in C_c(W)$.

$$\begin{aligned} \iint_W p(t, \mathbf{x}) \partial_t \psi(t, \mathbf{x}) d\mathbf{x} dt + \iint_W p(t, \mathbf{x}) \nabla \psi(t, \mathbf{x}) d\mathbf{x} dt + \\ + \int_{\Omega} p(0, \mathbf{x}) \psi(0, \mathbf{x}) d\mathbf{x} = 0. \end{aligned} \quad (5)$$

Choose $\psi \in C_c((0, \infty) \times \Omega)$. Then (5) becomes

$$\iint_W p(t, \mathbf{x}) \partial_t \psi(t, \mathbf{x}) d\mathbf{x} dt + \iint_W p(t, \mathbf{x}) \nabla \psi(t, \mathbf{x}) d\mathbf{x} dt = 0.$$

Let $\mathbf{y} = \phi_t(x)$ and $\eta_t(x) = \det(\partial_x \Phi_t(x))$. Also let $W_t = \Phi_t(W)$. Then, the first integral becomes

$$\iint_{W_t} e^{Q(\Phi_t(\mathbf{y})) - Q(\mathbf{y})} p_0(\mathbf{y}) \partial_t \psi(t, \Phi_t(\mathbf{y})) \eta_t(\mathbf{y}) d\mathbf{y} dt.$$

The second integral becomes

$$\iint_{W_t} e^{Q(\Phi_t(\mathbf{y})) - Q(\mathbf{y})} p_0(\mathbf{y}) \nabla \cdot \psi(t, \Phi_t(\mathbf{y})) \eta_t(\mathbf{y}) d\mathbf{y} dt.$$

Combining both integrals, we can write

$$\iint_{W_t} e^{Q(\Phi_t(\mathbf{y})) - Q(\mathbf{y})} p_0(\mathbf{y}) \frac{d}{dt} \psi(t, \Phi_t(\mathbf{y})) \eta_t(\mathbf{y}) d\mathbf{y} dt.$$

On integrating by parts, we have that

$$\iint_{W_t} e^{-Q(\mathbf{y})} p_0(\mathbf{y}) \psi(t, \Phi_t(\mathbf{y})) e^{Q(\Phi_t(\mathbf{y}))} [F \cdot \nabla Q - \nabla \cdot F] d\mathbf{y} dt = 0.$$

We have that p is clearly nonnegative. Moreover, Let $V = \text{supp}(u(t, \cdot))$ and let $V_t = (\Phi_{-t}(W))$. Let ψ be a vanishing function in Ω . Then

$$0 = \int_{\Phi_t(\Omega)} u(t, \mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \int_{\Omega} e^{Q(\Phi_t(\mathbf{y})) - Q(\mathbf{y})} u_0(\mathbf{y}) \psi(\mathbf{y}) \text{div}(F) d\mathbf{y}$$

Hence $\text{supp}(u(t, \cdot)) \subset \Phi_t(\Omega)$. Since Ω is regularly attracting for F , we have that $\Phi_t(\Omega) \subset \Omega$. If $\text{supp}(p_0(\mathbf{y})) \subset \Omega$, then we extend p_0 by defining it to be

zero in $\mathbb{R}^N - U$. Then we have that

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} u(t, \cdot) d\mathbf{x} &= \frac{d}{dt} \int_V u(t, \cdot) d\mathbf{x} \\ &= - \int_V \operatorname{div}(u(t, \cdot) \hat{F}) d\mathbf{x} = 0.\end{aligned}$$

□

Let

$$F_S(x, y) = (c(1 - x - y) - \alpha xy, \alpha xy - by).$$

We define the reflected SIR field by

$$F_{RS}(x, y) = (c(1 - x - y) - \alpha xy, (ax - b)|y|)$$

We immediately have

Lemma 1. *Let Ω be the simplex in the nonnegative orthant of \mathbb{R}^2 . Then F_{RS} is a C^1 vector field in \mathbb{R}^2 , and Ω is regularly attracting for F_{RS} .*

Thus for smooth solutions, we have

Theorem 2. *Consider the Cauchy problem for (1), with a L^1 non-negative initial condition p_0 . Then there exists a unique solution satisfying (2). Moreover, $p(t, x) \geq 0$.*

4. Asymptotic Behavior and measure solutions

We recall that the dynamics of SIR are controlled by the parameter $R_0 := a/b$ (see [2]), i.e,

Proposition 1. *Let $\mathbf{x}_1 = (1, 0)$, and $\mathbf{x}_2 = \left(\frac{1}{R_0}, \frac{c}{c+b} \left(1 - \frac{1}{R_0}\right)\right)$ be the two equilibria of SIR. \mathbf{x}_1 is referred to as the disease free equilibrium and \mathbf{x}_2 as the endemic equilibrium. If $R_0 \leq 1$, then any solution that starts in the nonnegative orthant of \mathbb{R}^2 approaches \mathbf{x}_0 for large time. If $R_0 > 1$, then \mathbf{x}_1 is the limiting point.*

With this point of view we have

Theorem 3. *Let p be a solution of (1), satisfying (2). Then, in the Wasserstein metric, we have that*

$$\lim_{t \rightarrow \infty} p(t, x) = \begin{cases} \delta_{\mathbf{x}_1}, & R_0 \leq 1 \\ \delta_{\mathbf{x}_2}, & R_0 > 1. \end{cases}$$

Proof. We deal with the case $R_0 \leq 1$; the case $R_0 > 1$ is analogous. Since $R_0 \leq 1$, we have that \mathbf{x}_1 is the globally asymptotic stable equilibrium. Then given, $\delta > 0$, we can find $T > 0$, such that, for $t > T$, we have that

$$\Phi_t(\mathcal{S}) \subset B_\delta(\mathbf{x}_1).$$

Let $\psi(\mathbf{x})$ be a continuous function. Then, for $t > T$, we have that

$$\int_{\mathcal{S}} p(t, \mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \int_{B_\delta(\mathbf{x}_1)} p(t, \mathbf{x}) \psi(\mathbf{x}) d\mathbf{x}.$$

But, let $\epsilon > 0$ be given. Since ψ is continuous, we have $\delta > 0$, such that

$$\psi(\mathbf{x}_1) - \epsilon \leq \int_{B_\delta(\mathbf{x}_1)} p(t, \mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} \leq \psi(\mathbf{x}_0) + \epsilon,$$

and this proves the claim, since we have that

$$\int_{B_\epsilon(x_1)} p(t, \Phi_t(\mathbf{y})) d\mathbf{y} = \int_{B_\epsilon(x_1)} e^{Q(\Phi_t(\mathbf{y})) - Q(\mathbf{y})} p_0(\mathbf{y}) \psi(\Phi_t(\mathbf{y})) \eta_t(\mathbf{y}) d\mathbf{y}$$

□

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